LIMIT CURRENTS IN A PLASMA BETATRON

A. M. Stefanovskii

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The radial motion of a plasma column is considered for electron acceleration in a plasma betatron. The limit value of the relativistic currents which can be obtained in devices of this type is computed.

G. I. Budker [1] proposed using the runaway effect in a plasma with a strong electric field for converting a cold ring plasma into an intense compensated beam of relativistic electrons. To confine such a beam within an annular vacuum chamber one can use either a betatron-type magnetic field or the field of the image currents produced in the metal shell enclosing the vacuum chamber with the electron beam. In the latter case, as estimates show, the number of accelerated electrons must already be considerable; this leads to an increase in the difficulties which impede the successful acceleration of all plasma electrons.

Accordingly, most of the experiments on accelerating plasma electrons have employed betatron fields in devices called plasma betatrons [2–4]. A feature of these accelerators is total compensation of the space charge of the accelerated electrons and hence an increased possibility of obtaining high accelerated currents. In this article, we compute the magnitude of the limit currents which may be obtained in a plasma betatron as a function of its parameters and operating conditions.

The first results in this direction, published in 1949 [5], were rough estimates. Subsequently, other more accurate calculations were published [6], but these, in our opinion, did not give sufficient information on the characteristic quantities.

In order to confine an electrically neutral electron beam within an annular vacuum chamber, it is necessary to compensate for radial forces tending to project the beam outward. There are several such forces: the centrifugal forces of the accelerated particles, the electrodynamic force associated with the magnetic energy of the beam as a whole, and, finally, the plasma gas pressure or, if a toroidal magnetic field is used, diamagnetic expulsion of the plasma. Let us consider the compensation for these forces by an external magnetic field, i.e., high-current betatron operation.

The equations of motion of the electrons and ions with respect to r and φ in a cylindrical coordinate system (the z axis coincides with the betatron axis) are as follows.

1) The equation of motion with respect to φ due to conservation of generalized momentum for electrons is

$$\gamma m r v_\varphi - \frac{e}{2\pi c}\pi r^2 H^\circ(t) + \frac{e n_1 (v_\varphi + V_\varphi)}{2\pi c} L = 0 \quad \left(n_1 = \frac{N}{2\pi r}\right), \tag{1}$$

where L is the plasma beam inductance, $H^0(t)$ is the average magnetic field within a circle of radius r, n_1 is the number of electrons (or ions) per unit beam length, v_φ and V_φ are the velocities of electrons and ions with respect to φ; the rest of the notation is as usual.

2) The equation of radial motion of an electron is

$$\gamma m \frac{d^2}{dt^2}\Delta r_e - 4\pi e^2 n(\Delta r_i - \Delta r_e) = \frac{\gamma m v_\varphi^2}{r} - \frac{e v_\varphi H(t)}{c} + \frac{1}{N}\frac{\partial W_m}{\partial r} + \frac{W_{te}}{r}. \tag{2}$$

Here Δr_e and Δr_1 are the deviations of the electron- and ion-beam centers from their initial position, H(t) is the external magnetic field at radius r, W_m is the beam magnetic energy, and W_{te} is a special form for the average energy of thermal motion of the electrons.

The analogous equations for ions have the form

$$M r V_\varphi - \frac{e}{2\pi c}\pi r^2 H^\circ(t) + \frac{e n_1 (v_\varphi + V_\varphi)}{2\pi c} L = 0, \tag{3}$$

$$M \frac{d^2}{dt^2}\Delta r_i + 4\pi e^2 n(\Delta r_i - \Delta r_e) = \frac{M V_\varphi^2}{r} - \frac{e V_\varphi H(t)}{c} + \frac{e V_\varphi H_e}{c} + \frac{W_{ti}}{r}. \tag{4}$$

Here H_e is the magnetic field of the electron beam.

It is clear from equation (3) that $V_\varphi = v_\varphi \gamma m/M$, i.e., the ion velocity with respect to φ in the nonrelativistic region is much less than the electron velocity; therefore it can be ignored in equation (1). Then

$$v_\varphi = \frac{erH^\circ(t)}{2\gamma mc}\left(1 + \frac{\nu l}{\gamma}\right)^{-1} \quad \left(\nu = \frac{e^2 n_1}{mc^2}, \quad l = \frac{L}{2\pi r}\right). \tag{5}$$

Here l is the inductance per unit length of plasma column. The last expression indicates that with increase in the number of accelerated electrons ($\nu \sim n_1$) their velocity increases more slowly than in an ordinary betatron. This is due to the fact that the external electric field increases not only the kinetic energy of the electrons but also the magnetic energy of the beam as a whole. Note also that the two different components that make up the column inductance have different effects on the electron velocity. One of these is associated with the beam external magnetic flux and attenuates the accelerating electric field of the betatron E in the same way for all electrons. The other part of the magnetic flux penetrates the core of the beam with the result that the field at the center of the beam section is smaller than at its edges. The effect of the external portion of the flux can, of course, be described by means of the multiplier $(1 + \nu l_e \gamma^{-1})^{-1}$.

The effect of the internal portion of the flux can be allowed for by using the velocity averaged over the beam cross section in place of the electron velocity.

At small values of ν ($\nu \ll 1$), the averaged velocity (or electric field) is equal to the electron velocity at the edge of the beam multiplied by $(1 - \nu\,[2\gamma]^{-1}]$. Therefore, the total change in velocity for small $\nu l \gamma^{-1}$ is described by expression (5) with $l = l_i + 1/2$.

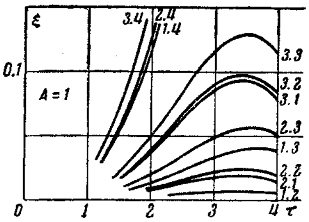

Fig. 1. Radial motion of plasma beam for A = 1. The first figures on the curves mean the following: 1) $B = 0.3 \cdot 10^{-2}$; 2) $B = 1.5 \cdot 10^{-2}$; 3) $B = 7 \cdot 10^{-2}$; the second figures mean: 1) $\Delta v_\varphi / c = 0$; 2) $\Delta v_\varphi / c = 10^{-3}$; 3) $\Delta v_\varphi / c = 10^{-2}$; 4) $\Delta v_\varphi / c = 10^{-1}$.

We now write an expression for the radical force acting on the electron beam in a betatron taking into account the intrinsic fields. In doing so, we note that $W_{te} r^{-1}$ is equal to $m v_{te}^2 r^{-1}$ in order of magnitude and for $v_\varphi \gg v_{te}$ is negligibly small in comparison with the centrifugal force $r^{-1} \gamma m v_\varphi^2$. Therefore it will be ignored in what follows. Moreover, the magnitude of the intrinsic magnetic force of the electron current is

$$\frac{1}{N} \frac{\partial W_m}{\partial r} = \frac{\nu l}{2\gamma} \left(1 + \frac{r}{l} \frac{\partial l}{\partial r}\right) \frac{\gamma m v_\varphi^2}{r}.$$

In this connection, the right side of equation (2) may be reduced to the form

$$F = \frac{\gamma m v_\varphi}{r} \left\{ v_\varphi \left[1 + \frac{\nu l}{2\gamma} \left(1 + \frac{r}{l} \frac{\partial l}{\partial r}\right)\right] - \frac{eHr}{\gamma mc} \right\}. \qquad (6)$$

Using (5), we see that the force F vanishes when

$$H = \frac{1}{2} H^\circ \left[1 + \frac{\nu l}{2\gamma} \left(1 + \frac{r}{l} \frac{\partial l}{\partial r}\right)\right] \left(1 + \frac{\nu l}{\gamma}\right)^{-1}. \qquad (7)$$

For $\nu l \gamma^{-1} \ll 1$ relationship (7) is transformed into the usual "two to one" rule for betatrons, while for $\nu l \gamma^{-1} \gg 1$ it reduces to the high-current formula obtained for a particular case by Osovets in [7].

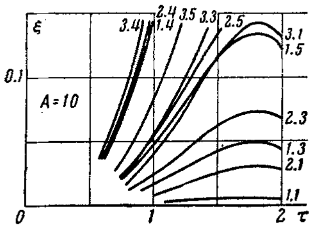

Fig. 2. Radial motion of plasma beam for A = 10. The first figures have the same meaning as in Fig. 1. The second figure 5 corresponds to $\Delta v_\varphi / c = 3 \cdot 10^{-2}$.

Thus, for motion of the electrons along a constant-radius orbit, the relationship between the accelerating field $E(\sim H^0)$ and the sustaining field H depends upon the number of accelerated electrons ν, their energy γ, and the beam geometry l in the accelerator. It is very difficult to satisfy relationship (7) for a wide range of values of ν, γ, and l.

As a result, it is of interest to investigate the radial motion of a beam of electrons in an ordinary betatron field. In this case, the "two to one" rule is satisfied on a circular orbit of radius R, the field H varies according to the usual law $H = H_0 (R/r) n_b$, and the width of the magnetic track is small in comparison with the radius R. Then, letting $H = H_0 (1 - n_b \xi)$, where $\xi = \Delta r / R$ and $H^0 = 2H_0 (1 - \xi)$, from (6) we obtain

$$F = \frac{\gamma v_\varphi e H(t)}{c} \left(1 + \frac{\nu l}{\gamma}\right)^{-1} \left\{ (1 - \xi) \left[1 + \frac{\nu l}{2\gamma} \left(1 + \frac{r}{l} \frac{\partial l}{\partial r}\right)\right] - (1 - n_b \xi) \left(1 + \frac{\nu l}{\gamma}\right) \right\}. \qquad (8)$$

The right side of this expression vanishes at

$$\xi = -\frac{1}{1 - n_b} \frac{\nu l}{2\gamma} \left(1 - \frac{r}{l} \frac{\partial l}{\partial r}\right) \qquad (9)$$

if we ignore terms containing $\xi\,(\nu l / \gamma)$.

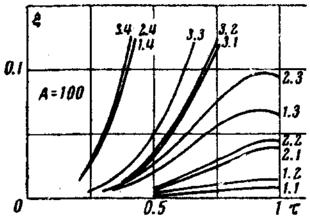

Fig. 3. Radial motion of plasma beam for A = 100. Same notation as in Fig. 2.

This expression determines the displacement of the equilibrium orbit of the electron beam from the equilibrium vacuum orbit in the betatron for different values of ν, l and γ. It is clear that this shift decreases in time as the energy (i.e., γ) increases, being at its maximum when acceleration begins. Substituting the maximum permissible value for the displacement, the halfwidth of the magnetic track Δ, into equation (9), we obtain the maximum value of ν for which at any moment of time the equilibrium orbit of the electron beam lies within the limits of the magnetic track.

$$\nu_{max} = (1 - n_b) \frac{2\Delta}{R} \left(l - r \frac{\partial l}{\partial r}\right)^{-1}. \qquad (10)$$

For $\Delta / R = 1/10$ ($R/a \approx 10$), $n_b = 0.5$ and $l = 2 \times (\ln(8R/a) - 1.75)$, ν_{max} is equal to $2 \cdot 10^{-2}$, which corresponds to a relativistic electron current of about 300 A.

The actual position of the electron beam equilibrium orbit in the betatron is not determined solely by the ratio of the intrinsic fields of the beam and the external fields of the betatron. Yet another factor has a significant effect on the position of the equilibrium orbit. It consists of the following.

It is known that when plasma electrons are accelerated various instabilities appear, as a result of which various plasma oscillations are excited. Here, the energy of the oscillating fields is taken from the energy of directed motion of the electrons, so that the electron velocity v_φ decreases by some value Δv_φ. We shall assume below that $\Delta v_\varphi/v_\varphi \ll 1$ in all the cases considered.

Electron-oscillation interaction takes place at velocities which do not exceed the maximum phase velocities of the plasma waves. Here, only potential plasma waves and waves associated with transverse beam deflection are considered, since it is precisely these oscillations which are the most dangerous [1]. The maximum phase velocity of the potential waves in an unbounded cylindrical plasma is of the order of $2c\sqrt{\nu}/2.4$, i.e., roughly speaking, approximately an order less than the speed of light in the conditions of interest ($\nu \sim 10^{-2}$). The maximum phase velocity of the spiral density waves in a plasma with a toroidal magnetic field H_φ is $4\pi enRC(H_\varphi)^{-1}$, which also comprises roughly 1/10 the speed of light under plasma betatron conditions. Thus, the appearance of a Δv_φ for electrons (which we shall refer to as the "detuning") is observed at the very beginning of acceleration, until the electrons acquire an energy of 5–10 keV. For the sake of simplicity, we shall assume that this detuning is present from the very beginning of acceleration. It will become clear that this assumption does not seriously affect the final results.

A new expression for the electron velocity is obtained from equation (1); in the presence of a detuning Δv_φ

$$v_\varphi = \frac{er\,H^\circ(t)}{2\gamma mc}\left(1+\frac{\nu l}{r}\right)^{-1} - \frac{r_0}{\gamma}\Delta v_\varphi, \tag{11}$$

where r_0 is the location of the center of the electron beam when detuning appears. Substituting into (2) for the betatron field, we obtain

$$\gamma m\frac{d^2\xi_e}{dt^2} - 4\pi e^2 n(\xi_i - \xi_e) = \tag{12}$$
$$= -\frac{e^2H_0^2(t)}{\gamma mc^2}\left[(1-n_b)\,\xi_e + \frac{\nu l}{2\gamma}\left(1-\frac{r}{l}\frac{\partial l}{\partial r}\right)\right] - \frac{eH_0(t)}{R}\frac{\Delta v_\varphi}{c}.$$

In order to analyze equation (12) further, it is necessary to clarify the relationship between the forces acting on the electron beam. Thus, when the electron beam is displaced with respect to the ion beam, a polarization field appears obstructing the further radial motion of the electrons. The separation of the ion and electron beam centers can easily be obtained from (12) for $\ddot{\xi}_e = 0$ and $\xi_i = 0$. For zero detuning, the shift is equal to

$$\frac{\Delta r}{a} = \frac{a}{R}\frac{\gamma^2-1}{\gamma^2}\frac{l}{8}\left(1-\frac{r}{l}\frac{\partial l}{\partial r}\right), \tag{13}$$

where a is the radius of the plasma column. Since $a/R \ll 1$ and

$$\frac{\gamma^2-1}{\gamma^2}\frac{l}{8}\left(1-\frac{r}{l}\frac{\partial l}{\partial r}\right) \lesssim 1$$

the beam separation turns out to be very small at all stages of acceleration, particularly in the nonrelativistic region when $\gamma^2 \sim 1$. This means that the electric polarization field acts on the beam much more strongly than does the magnetic field of the betatron. In addition, it turns out that the beam separation does not depend upon the beam density, which is also quite natural. Analysis shows that Eq. (13) also remains valid for small electron velocity shifts.

Thus, the betatron forces acting on the electron beam through the polarization field are applied to the ions. After a number of formal operations, the equations of radial motion of ions and electrons are obtained in the form

$$M\ddot{\xi}_i = -\frac{e^2H_0^2(t)}{\gamma mc^2}\left[(1-n_b)\,\xi_i + \frac{\nu l}{2\gamma}\left(1-\frac{r}{l}\frac{\partial l}{\partial r}\right)\right] - \frac{eH_0(t)\,\Delta v_\varphi}{Rc}, \tag{14}$$

$$\gamma m\,(\Delta\ddot{\xi}_e + \omega_0^2\Delta\xi_e) = 0. \tag{15}$$

Electron oscillations take place about the equilibrium position $\xi_{e0} = \xi_i + \Delta\xi_{e0}$, where $\Delta\xi_{e0}$ is determined from Eq. (13).

The frequency of these oscillations almost coincides with the plasma frequency, while the amplitude is equal to $\Delta\xi_{e0}$.

The total radial motion of the plasma column is given by Eq. (14). We introduce a new variable and certain notation:

$$\tau = \frac{eEct}{mc^2} = \frac{eH_0^{\cdot}(t)\,R}{mc^2},\quad A = \frac{m}{M}\left(\frac{mc^2}{eER}\right)^2,$$

$$B = \frac{\nu l}{2}\left(1-\frac{r}{l}\frac{\partial l}{\partial r}\right).$$

Then Eq. (14) is written in the form

$$\ddot{\xi} + A\,(1-n_b)\frac{\tau^2}{\sqrt{1+\tau^2}}\,\xi = -AB\frac{\tau^2}{1+\tau^2} - A\tau\frac{\Delta v_\varphi}{c}. \tag{16}$$

Figures 1–3 give numerical solutions of Eq. (16) for A = 100, 10, and 1, which corresponds roughly to the acceleration of argon-plasma electrons in fields of 10, 30, and 100 V/cm. The values $B = 7\cdot10^{-2}$, $1.5\cdot10^{-2}$, and $0.3\cdot10^{-2}$ correspond to relativistic currents of 700, 170, and 35 A. The radius of the vacuum chamber is taken to be 20 cm, $n_b = 0.5$, and $l = 2\,(\ln(8R/a) - 1.75) \simeq 5.1$. The magnitude of Δv_φ is shown in Figs. 1–3. The initial position of the beam is taken to coincide with the vacuum betatron orbit.

As is clear from Figs. 1–3, the escape of electrons to the walls of the vacuum chamber as a result of detuning for A = 1 and 10 takes place at energies above 150 keV. This is much greater than the energy at which the electrons interact with the oscillations. Therefore, in the cases mentioned, the assumption about detuning being present from the very beginning of acceleration is completely justified. For A = 100, the energy of electrons escaping to the wall is 30–60 keV. Evidently, under actual conditions, with these parameters the escape of electrons takes place somewhat later than indicated by Eq. (16).

The results of integrating Eq. (16) show that allowance for ion inertia leads to a significant increase in the maximum current in the plasma betatron.

This is explained by the fact that the equilibrium orbit of the beam approaches the vacuum orbit ever more closely with increase in electron energy. At the same time, the ions cannot move rapidly enough owing to their greater mass. While the necessary ion shift takes place, the electron equilibrium orbit can approach quite close to the vacuum equilibrium orbit.

From the form of the curves in Figs. 1–3 we can draw the conclusion that the equilibrium orbit of the plasma beam is stable for the selected parameters. Analysis of Eq. (12) also leads to a similar conclusion.

We give an expression for n_b ensuring plasma beam stability with respect to both r and z, omitting the simple computations:

$$\left[1 + \frac{\nu l}{\gamma}\left(1 - \frac{r^2}{l}\frac{\partial^2 l}{\partial r^2}\right)\right]\left[1 + \frac{\nu l}{2\gamma}\left(1 + \frac{r}{l}\frac{\partial l}{\partial r}\right)\right]^{-1} < \qquad (17)$$

$$< n_b < 0\ .$$

In conclusion we note the following. The electron-beam inductance is assumed to be constant during the entire acceleration cycle. Actually, this assumption is only valid when a sufficiently strong toroidal magnetic field H_φ is used (of the order of 500 Oe). Here, if

$$4\pi n M c_2 H_\varphi^{-2} \gg 1$$

(which always holds in actual conditions), the equations of radial motion of the ions do not change; therefore the obtained results will also be valid for betatrons with a longitudinal magnetic field. The electron motion changes somewhat: instead of oscillations with respect to r about the equilibrium position at the plasma frequency, we get rotation in the zr plane about the point $\xi_i + \Delta\xi_{e0}$ at the frequency $4\pi encH_\varphi^{-1}$.

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